Homotopy Algebras via Resolutions of Operads

Martin Markl*

All algebraic objects in this note will be considered over a fixed field **k** of characteristic zero. If not stated otherwise, all operads live in the category of differential graded vector spaces over **k**. For standard terminology concerning operads, algebras over operads, etc., see either the original paper by J.P. May [May72], or an overview [Lod].

The aim of this note is mainly to advocate our approach to homotopy algebras based on the minimal model of an operad. Our intention is to expand it to a paper on homotopy properties of the category of homotopy algebras; some possible results in this direction are indicated in Section 3.

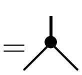
I would like to express my thanks to Rainer Vogt for his kind invitation to Osnabrück. I also owe my thanks to Jim Stasheff for careful reading the manuscript and many useful comments.

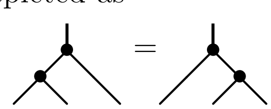
1. Motivations.

Example 1. An associative algebra consists of a vector space A together with a multiplication $\mu : A \otimes A \to A$, which is supposed to be associative:

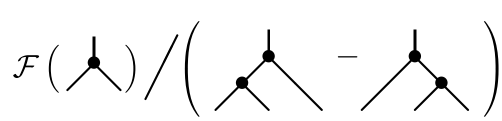
$$\mu(\mu(a,b),c) = \mu(a,\mu(b,c));$$

we neglect the role of a unit.

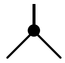
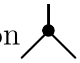
We can 'visualize' μ as an 'operation' with two inputs and one output, $\mu =$ . The associativity can be then depicted as

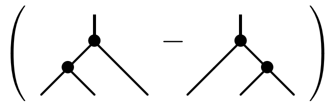


This means that the operad $\mathcal{A}ss$ describing associative algebras in the category of differential graded vector spaces can be presented as the quotient

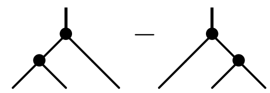


*The author was supported by the grant GA AV ČR 1019804. This paper is in final form and no version of it will be submitted for publication elsewhere.

where $\mathcal{F}\Big($  $\Big)$ is the free non-Σ operad on the operation  representing the product μ and



is the operadic ideal generated by the element



corresponding to the associativity axiom. As differential graded vector spaces, the pieces $Ass(n)$, $n \geq 1$, of the operad Ass are concentrated in degree zero and have trivial differentials.

Homotopy versions of associative algebras are so-called A(∞)-algebras (also called strongly homotopy associative algebras), introduced by J. Stasheff [Sta63]. An A(∞)-algebra $A = (A, \partial, \mu_2, \mu_3, \ldots)$ consists of a differential graded vector space $A = (A, \partial)$ and multilinear operations $\mu_n : A^{\otimes n} \to A$ of degree $n - 2$ that satisfy the following infinite set of axioms:

$$
\begin{aligned}
(1) \qquad \mu_2(\mu_2(a,b)c) - \mu_2(a, \mu_2(b,c)) &= [\mu_3, \partial](a,b,c),\\
\mu_3(\mu_2(a,b),c,d) - \mu_3(a, \mu_2(b,c),d) + \mu_3(a,b,\mu_2(c,d)) - &\\
-\mu_2(\mu_3(a,b,c),d) - (-1)^{|a|}\cdot\mu_2(a, \mu_3(b,c,d)) &= [\mu_4, \partial](a,b,c,d),\\
&\ \vdots\\
\sum_{\substack{i+j=n+1\\ i,j\geq 2}} \sum_{s=0}^{n-j} (-1)^{\epsilon}\cdot\mu_i(a_1, \ldots, a_s, \mu_j(a_{s+1}, \ldots, a_{s+j}), a_{s+j+1}, \ldots, a_n) &= [\mu_n, \partial](a_1, \ldots, a_n),
\end{aligned}
$$

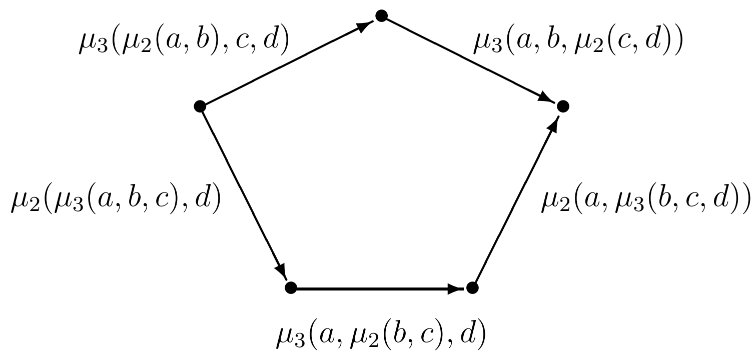
where $[\mu_n, \partial](a_1, \ldots, a_n)$ denotes the induced differential in the endomorphism complex $\mathrm{Hom}_{\mathbf{k}}(A^{\otimes n}, A)$,

$$
\begin{aligned}
&[\mu_n, \partial](a_1, \ldots, a_n) :=\\
&\qquad := \sum_{1\leq s\leq n} (-1)^{|a_1|+\cdots+|a_{s-1}|}\cdot\mu_n(a_1, \ldots, \partial a_s, \ldots, a_n) - (-1)^n\cdot\partial\mu_n(a_1, \ldots, a_n).
\end{aligned}
$$

The sign is given by

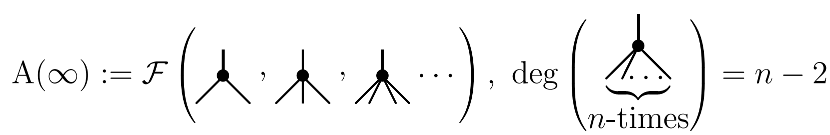
$$
\epsilon := j + s(j+1) + j(|a_1| + \cdots + |a_{s-1}|).
$$

The first axiom of (1) expresses the homotopy associativity of the product μ_2. The second axiom is a linearized version of the Mac Lane pentagon condition, with the terms in the left hand side corresponding to the edges of the pentagon:

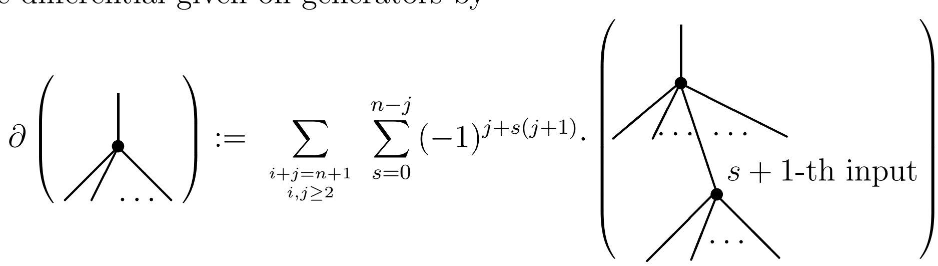


This miraculous phenomenon is explained by the fact that the operad A(∞) is isomorphic to the operad $\{CC_*(K_n)\}_{n\geq 1}$ of cellular chains of a certain cellular operad $\{K_n\}_{n\geq 1}$, introduced by J. Stasheff in his pioneering work [Sta63]. The spaces K_n are now called associahedra; the polyhedron K_4 that parameterizes 4-ary operations is the pentagon mentioned above.

A more straightforward description of A(∞) can be obtained from the axioms as follows. The differential graded operad A(∞) is free as an operad,



with the differential given on generators by

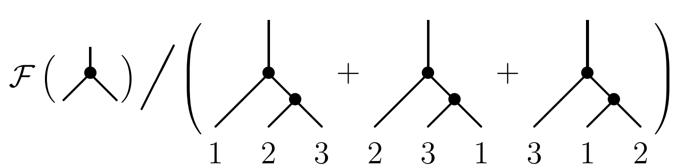


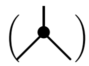
and extended by the derivation property. A(∞)-algebras occur in Nature as chain algebras of loop spaces, a result which is today classical, see again [Sta63].

Example 2. A Lie algebra is a vector space L with an antisymmetric product $[-,-] : L \otimes L \to L$ (the 'bracket'), satisfying the Jacoby identity:

$$[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0.$$

Thus the operad *Lie* for Lie algebras is the quotient



Here $\mathcal{F}$  now denotes the free *symmetric* (Σ-) operad on one antisymmetric bilinear operation; the labels of the inputs encode the action of the symmetric group in a standard manner.

Homotopy versions of Lie algebras are strongly homotopy Lie algebras (also called sh Lie algebras or L(∞)-algebras). They were introduced and systematically studied in [LS93], though they had existed in the literature, in various disguises, even before; see also [LM95].

An L(∞) algebra is a differential graded vector space $L = (L, \partial)$, together with a set $\{l_n\}_{n\geq 2}$ of graded antisymmetric operations $l_n : L^{\otimes n} \to L$ of degree $n-2$ such that the following infinite set of axioms is satisfied for any $n \geq 2$:

$$\sum_{\substack{i+j=n+1\\ i,j\geq 2}} \sum_{\sigma} \chi(\sigma)(-1)^{i(j-1)} l_j(l_i(a_{\sigma(1)}, \dots, a_{\sigma(i)}), a_{\sigma(i+1)}, \dots, a_{\sigma(n)}) =$$

$$= (-1)^n \cdot [l_n, \partial](a_1, \dots, a_n)$$

The summation is taken over all $(i, n-i)$-unshuffles

$$\sigma \in \Sigma_n,\ \sigma(1) < \cdots < \sigma(i),\ \sigma(i+1) < \cdots < \sigma(n),$$

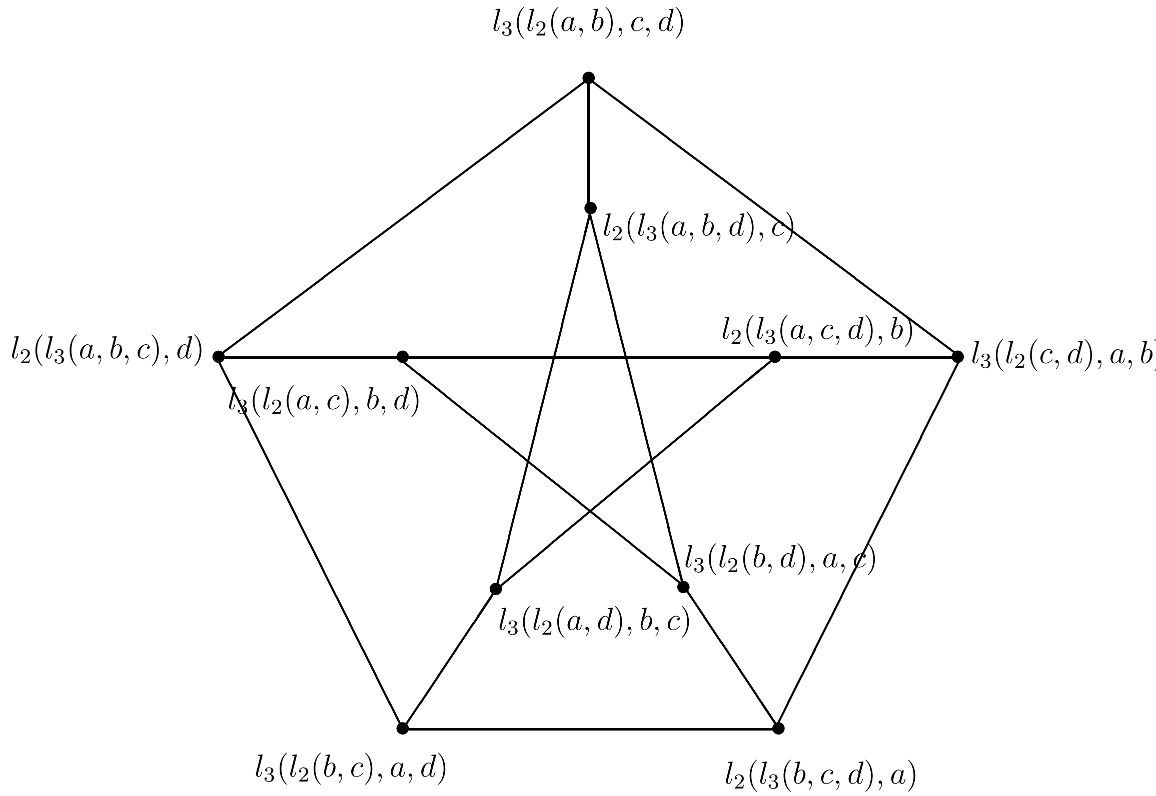
with $n-1 \geq i \geq 1$, and $\chi(\sigma)$ is a certain sign which we will not specify here, see [LM95]. We write the first two axioms explicitly (though, to save paper, without signs). The first axiom says that the 'bracket' l_2 satisfies the Jacobi identity up to a homotopy:

$$l_2(l_2(a,b),c) + l_2(l_2(b,c),a) + l_2(l_2(c,a),b) = [l_3, \partial](a,b,c).$$

The second condition reads

$$\begin{aligned} &l_2(l_3(a,b,c),d) + l_2(l_3(a,c,d),b) + l_2(l_3(a,b,d),c) + l_2(l_3(b,c,d),a) + \\ &\qquad + l_3(l_2(a,b),c,d) + l_3(l_2(a,c),b,d) + + l_3(l_2(a,d),b,c) + \\ &\qquad + l_3(l_2(b,c),a,d) + l_3(l_2(c,d),a,b) + l_3(l_2(b,d),a,c) = [\partial, l_4](a,b,c,d). \end{aligned}$$

To indicate an interpretation of the last equality, recall that there is a Lie-analog of K_4, which we introduced in [MS97]. We denoted it by L_4 and called the Lie-hedron. It is not a polyhedron, but just a graph, and the 10 terms in the left hand side of the above equation correspond to the vertices (not edges!) of L_4, which is the Peterson graph:



As in Example 1, the operad $\mathrm{L}(\infty)$ for strongly homotopy Lie algebras is a differential graded operad, which is free as an operad,

$$\mathrm{L}(\infty) := \mathcal{F}\left(\;,\;,\; \cdots \right), \quad \underbrace{\;\ldots\;}_{n\text{-times}} \text{ antisymmetric of degree } n-2,$$

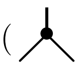
where $\mathcal{F}(-)$ is the free *symmetric* (Σ-) operad. The action of the differential is given by

$$(-1)^n \cdot \partial\left(\;\ldots\; \right) := \sum_{\substack{i+j=n+1\\ i,j\geq 2}} \sum_{\sigma} \chi(\sigma)(-1)^{i(j-1)} \cdot \left(\sigma(1)\,\sigma(2) \ldots \sigma(i)\;\sigma(i+1) \ldots \sigma(n) \right)$$

There is a stunning example of a strongly homotopy Lie algebra taken from Nature – the convolution product of functionals on the space of closed strings, see [SZ89, Sta92]. Another place where strongly homotopy Lie algebras naturally occur is the space of horizontal forms on the infinite jet bundle, where they appear as lifts of brackets on the space of Lagrangian functionals, see [BFLS97, MS98].

Homotopy versions of commutative associative algebras were introduced in [Kad85] by T. Kadeishvili. They are called C(∞), commutative, or balanced A(∞)-algebras (compare also [Mar92]).

Let us define a map $\alpha_{Ass} : \mathrm{A}(\infty) \to Ass$ by

$\alpha_{Ass}($  $) =$ the multiplication $\mu \in Ass(2)$,

while α_{Ass} is trivial on the remaining generators of $\mathrm{A}(\infty)$. We recommend as an exercise to verify that $\alpha_{Ass} \circ \partial = 0$, which means that α_{Ass} is a map of differential operads, if we interpret Ass as a differential operad with trivial differential. It can be shown that α_{Ass} induces an isomorphism of homology, $H_*(\mathrm{A}(\infty), \partial) \cong Ass$. There is also a map $\alpha_{Lie} : \mathrm{L}(\infty) \to Lie$ having similar properties.

2. Concepts.

Let us try to sum up common features of the above examples. In both cases, homotopy algebras were algebras over a differential graded operad which was free as an operad. Moreover, we had natural maps $\alpha_{Ass} : (\mathrm{A}(\infty), \partial) \to (Ass, 0)$ and $\alpha_{Lie} : (\mathrm{L}(\infty), \partial) \to (Lie, 0)$ inducing isomorphisms in homology.

Recall that, in [Mar96], we formulated the following definition.

Definition 3. *Let $\mathcal{P}$ be an operad in the category of differential graded vector spaces. A minimal model of $\mathcal{P}$ is a differential graded operad $\mathcal{M}_{\mathcal{P}} = (\mathcal{F}(E), \partial)$, where $\mathcal{F}(E)$ is the free operad on a collection E, together with a map $\alpha_{\mathcal{P}} : \mathcal{M}_{\mathcal{P}} \to \mathcal{P}$ that is a homology isomorphism. The minimality means that we assume that $\partial(E)$ consists of decomposable elements of the free operad $\mathcal{F}(E)$.*

We proved, for each differential graded operad $\mathcal{P}$, the existence and a kind of uniqueness of the minimal model. When $\mathcal{P}$ has trivial differential (which was the case of all our examples), the minimal model is in fact bigraded, but we will not use this property. Having in mind the above examples, we proposed, in [Mar96], the following definition.

Definition 4. *A homotopy $\mathcal{P}$-algebra is a differential graded vector space $A = (A, \partial)$ with an action of the minimal model $\mathcal{M}_{\mathcal{P}}$ of $\mathcal{P}$, in other words, a differential graded algebra over the operad $\mathcal{M}_{\mathcal{P}}$.*

For so called Koszul quadratic operads, the minimal model coincides with the cobar construction on the quadratic dual of the operad, and our definition agrees with the one given by V. Ginzburg and A. Kapranov in [GK94].

Our approach keeps the symmetric group action (i.e., on the level of algebras, the commutativity, anticommutativity, etc.) strict, so we cannot handle algebraic variants of homotopy-everything operads such as those considered in [Smi82, Smi94], nor various other homotopy algebras where the symmetry is relaxed up to a homotopy – like homotopy Gerstenhaber algebras in the sense of M. Gerstenhaber and A.A. Voronov [GV95].

In the following section we try to indicate what must be done to justify our approach.

3. Perspectives.

There exists a natural concept of a map $\mathbf{f} : A \to B$ of homotopy $\mathcal{P}$-algebras. It is a system of degree zero multilinear maps $\mathbf{f} = \{f_n : A^{\otimes n} \to B\}_{n\geq 1}$ satisfying a certain set of axioms. One of the axioms postulates that $f_1 : (A, \partial) \to (B, \partial)$ is a morphism of the underlying differential graded vector spaces. Let $h\mathcal{P}$ denote the category of homotopy $\mathcal{P}$-algebras in the sense of Definition 4 and their maps.

Definition 5. *Let A and B be two homotopy $\mathcal{P}$-algebras and $g : (A, \partial) \to (B, \partial)$ a map of underlying differential graded vector spaces. A homotopy $\mathcal{P}$-structure on g is an $h\mathcal{P}$-map $\mathbf{f} : A \to B$ such that $f_1 = g$.*

In order to justify the notion of homotopy $\mathcal{P}$-algebras and their maps, we need to show that homotopy $\mathcal{P}$-structures are 'stable under a homotopy.' The precise meaning of this was explained, for topological spaces, in Chapter 1 of [BV73]. Corresponding statements translated to our algebraic language are the following.

1. For each homotopy $\mathcal{P}$-algebra A, differential graded vector space $B = (B, \partial)$ and a map $g : (A, \partial) \to (B, \partial)$ that is a homology isomorphism, there exist a homotopy $\mathcal{P}$-structure on (B, ∂) and a homotopy $\mathcal{P}$-structure $\mathbf{f} : A \to B$ on g.
2. Suppose A and B are two homotopy $\mathcal{P}$-algebras and $\mathbf{f} : A \to B$ a homotopy $\mathcal{P}$-algebra map. Suppose that $g : (A, \partial) \to (B, \partial)$ is a differential map that induces the same map of homology as f_1. Then there exists a homotopy $\mathcal{P}$-structure on g.
3. Suppose that $\mathbf{f} : A \to B$ is a homotopy $\mathcal{P}$-algebra map such that f_1 is a homology isomorphism. Suppose that $g : (B, \partial) \to (A, \partial)$ is a homology inverse of f_1. Then there exists a homotopy $\mathcal{P}$-structure on g.

We recommend, as an easy exercise, to prove that conditions 1 and 2 imply

1′. For each homotopy $\mathcal{P}$-algebra B, differential graded vector space $A = (A, \partial)$ and a map $g : (A, \partial) \to (B, \partial)$ that is a homology isomorphism, there exist a homotopy $\mathcal{P}$-structure on (A, ∂) and a homotopy $\mathcal{P}$-structure $\mathbf{f} : A \to B$ on g.

As far as we know, nobody has considered the above properties in full generality, though there are several partial results in this direction. Let us quote at least the following theorem due to T. Kadeishvili [Kad80, p. 232], which is a special case of 1′ for the category of A(∞)-algebras, with $(A, \partial) = (H(C), \partial = 0)$ and $B = (C, \partial, \mu_2, 0, \ldots)$.

Theorem 6. *Let (C, ∂) be a chain algebra. Then there exists an* A(∞)*-structure $\{X_k; k \geq 2\}$ on the graded space $H(C)$ having $X_2(a, b) = a \cdot b$ (the multiplication induced by μ_2), together with an* A(∞)*-homomorphism $\mathbf{f} : (H(C), 0, X_2, X_3, \ldots) \to (C, \partial, \mu_2, 0, \ldots)$ such that $f_1 : H(C) \to C$ is a homology isomorphism.*

A similar statement for balanced A(∞)-algebras was proved in [Mar92] by the author. Another direction of results that indicates a certain homotopy stability of homotopy algebraic structures is represented by various versions of the Perturbation Lemma, see [GS86, HK91].

References.

[BFLS97] G. Barnich, R. Fulp, T. Lada, and J.D. Stasheff. The sh Lie structure of Poisson brackets in field theory. Preprint, 1997.

[BV73] J.M. Boardman and R.M. Vogt. *Homotopy Invariant Algebraic Structures on Topological Spaces*. Springer-Verlag, 1973.

[GK94] V. Ginzburg and M.M. Kapranov. Koszul duality for operads. *Duke Math. Journal*, 76(1):203–272, 1994.

[GV95] M. Gerstenhaber and A.A. Voronov. Vyschie operacii na komplekse Hochschilda. *Fukcion. Analiz i ego Pril.*, 29(1):1–6, 1995. In Russian.

[GS86] V.K.A.M. Gugenheim and J.D. Stasheff. On perturbations and A_∞-structures. In L. Lemaire, editor, *Festschrift in honor of G. Hirsch's 60'th birthday*, volume 38 of *Bull. Soc. Math. Belgique*, pages 237–245, 1986.

[HK91] J. Huebschmann and T. Kadeishvili. Small models for chain algebras. *Math. Z.*, 207:245–280, 1991.

[Kad80] T.V. Kadeishvili. On the homology theory of fibre spaces. *Russian Math. Surveys*, 35:231–238, 1980.

[Kad85] T.V. Kadeishvili. O kategorii differentialnych koalgebr i kategorii $A(\infty)$-algebr. *Trudy Tbilisskogo Matematiceskogo Instituta*, pages 50–70, 1985. In Russian.

[LM95] T. Lada and M. Markl. Strongly homotopy Lie algebras. *Communications in Algebra*, 23(6):2147–2161, 1995.

[Lod] J.-L. Loday. La renaissance des opérades. *Séminarie Bourbaki*, 792. 47ème année 1994-95.

[LS93] T. Lada and J.D. Stasheff. Introduction to sh Lie algebras for physicists. *International Journal of Theoretical Physics*, 32(7):1087–1103, 1993.

[Mar92] M. Markl. A cohomology theory for $A(m)$-algebras and applications. *Journ. Pure Appl. Algebra*, 83:141–175, 1992.

[Mar96] M. Markl. Models for operads. *Communications in Algebra*, 24(4):1471–1500, 1996.

[May72] J.P. May. *The Geometry of Iterated Loop Spaces*, volume 271 of *Lecture Notes in Mathematics*. Springer-Verlag, 1972.

[MS97] M. Markl and S. Shnider. Coherence without commutative diagrams, Liehedra and other curiosities. Preprint q-alg/9712027, December 1997.

[MS98] M. Markl and S. Shnider. Differential operator endomorphisms of an Euler-Lagrange complex. Submitted, June 1998.

[SZ89] M. Saadi and B. Zwiebach. Closed string field theory from polyhedra. *Annals of Physics*, 192:213–227, 1989.

[Smi82] V.A. Smirnov. On the cochain complex of topological spaces. *Math. USSR Sbornik*, 43:133–144, 1982.

[Smi94] J.R. Smith. *Iterating the Cobar Construction*, volume 109 of *Memoirs of the AMS*. American Mathematical Society, 1994.

[Sta63] J.D. Stasheff. Homotopy associativity of H-spaces I,II. *Trans. Amer. Math. Soc.*, 108:275–312, 1963.

[Sta92] J. Stasheff. Towards a closed string field theory: Topology and convolution algebra. *Topology Proceedings*, 17:371–386, 1992.

Mathematical Institute of the Academy, Žitná 25, 115 67 Praha 1, Czech Republic,
email: `markl@math.cas.cz`